ETMAG LECTURE 3

Real numbers

Sequences of real numbers

Properties of the set of real numbers $\mathbb{R}$

Roughly speaking, the set of real numbers, $\mathbb{R}$, is the set of lengths of all possible segments PQ (including degenerate segments of the form PP) and their negatives. Obviously, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are (proper) subsets of $\mathbb{R}$. Elements of the set $\mathbb{R} \setminus \mathbb{Q}$ are called *irrational numbers.*

For every two different rational numbers a,b such that $a < b$ there is a rational number c such that $a<c<b$ (between every two rationals there is a third one). This property is called "being dense". So, $\mathbb{Q}$ is *dense*. Obviously $\mathbb{R}$ is also dense, $\mathbb{N}$ and $\mathbb{Z}$ are not. $\mathbb{R}\setminus\mathbb{Q}$ is also dense. Also, between every to different rational numbers there is an irrational one and vice versa.

Definition.

Every subset of $\mathbb{R}$ of the form (a;b) = $\{x \in \mathbb{R} | a < x\ and\ x < b\}$ is called an *open interval*. We allow b = ∞ and/or a = $-\infty$. A *closed interval* is a subset of $\mathbb{R}$ of the form [a;b] = $\{x \in \mathbb{R} | a \leq x\ and\ x \leq b\}$. We do NOT allow neither a nor b to be infinite, as symbols ∞ and $-\infty$ are not real numbers.

The above definitions extend easily to [a;b) and (a;b].

(1;1), (2;2], [0;0) are empty. [1;1] = $\{1\}$. If a>b then each of $(a; b), (a; b], [a; b),$ [a; b] is empty.

In every nonempty open interval there are rational and irrational numbers.

Absolute value

For every real number x,

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

Clearly, for every real numbers x, y, a we have

1. $0 \leq |x|$
2. $-|x| \leq x \leq |x|$
3. $|x| < a$ iff $-a < x < a$
4. $|x + y| \leq |x| + |y|$ (from 2., $-|x| - |y| \leq x + y \leq |x| + |y|$ which, by 3., implies $|x + y| \leq |x| + |y|$)

Bounded and unbounded sets

Definition A set T, $T \subseteq \mathbb{R}$ is said to be:

- *bounded from above* iff there exists a number M (**number!**) such that $(\forall x \in T) x \leq M$. *Every such* M is called an *upper bound* of T.
- *bounded from below* iff there exists a number K (a *lower bound*) such that $(\forall x \in T) K \leq x$
- *bounded* iff T is bounded from above AND from below.

Examples

- Every closed interval $[a;b]$ is bounded (from above by b and by every number $M \geq b$, from below by a and by every number $K \leq a$.
- Some open intervals are unbounded, e.g. $(-\infty; 5)$.
- $\mathbb{Z}$ is unbounded, $\mathbb{N}$ is bounded from below by anything ≤ 1, unbounded from above.
- $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is bounded from below by 0, from above by 1.
- $\emptyset$ is bounded by every real number both from above and from below.

Definition.

A number s is called the *least upper bound* or the *supremum* of a set $A \subseteq \mathbb{R}$ iff s is the smallest of all upper bounds of A. The least upper bound of A is denoted by $sup(A)$.

A number t is called the *greatest lower bound* or the *infimum* of a set $B \subseteq \mathbb{R}$ iff t is the largest of all lower bounds of B. The greatest lower bound of B is denoted by $inf(B)$.

- Clearly, not every subset of $\mathbb{R}$ has supremum or infimum.
- $inf(A)$, even if it exists, may belong or not belong to A. The same for $sup(A)$.

Theorem.

Every nonempty subset of $\mathbb{R}$ bounded from above has the (unique) least upper bound. Every nonempty subset of $\mathbb{R}$ bounded from below has the (unique) greatest lower bound.

Examples

- $sup\ [a;b] = sup\ (a;b] = sup\ (a;b) = sup\ [a;b) = b$. In each case infimum is a
- $sup(-\infty; 5) = 5$, $inf\ (-\infty; 5)$ does not exist.
- $sup\{a\} = inf\{a\} = a$.
- $inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$, from above by $sup\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 1$.
- $sup(\emptyset)$ and $inf\ (\emptyset)$ do not exist.

Sequences

Definition.
A *sequence* of real numbers is any function $a\colon \mathbb{N} \to \mathbb{R}$.

By ancient tradition instead of *a*(1), *a*(2), …, *a*(n), … etc we write $a_1, a_2, \dots, a_n, \dots$ and we call the numbers $a_1, a_2, \dots$ etc. *terms* rather than *values* of the sequence.

Often, we define a sequence writing (a_n) or $(a_n)_{n=1,2,\dots}$ or $(a_n)_{n=1}^{\infty}$ where a_n is a formula for calculating *n*-th term of the sequence.
Sometimes we write $(a_1, a_2, \dots, a_n, \dots)$, where the trailing ellipsis indicates that the sequence continues forever. Note that $(a_1, a_2, \dots, a_n)$ means a final, *n*-long sequence. For example, (*n*)=(1,2,…,n,…) denotes the sequence of natural numbers in their natural order, $\left(\frac{1}{n}\right) = (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots)$ means the sequence where for each *n*, $a_n = \frac{1}{n}$. (2^n) denotes the sequence of powers of 2 etc.

Definition.
A *sequence* (a_n) is said to be

- *increasing* iff $(\forall n)(a_n < a_{n+1})$
- *nondecreasing* iff $(\forall n)(a_n \leq a_{n+1})$
- *decreasing* iff $(\forall n)(a_n > a_{n+1})$
- *nonincreasing* iff $(\forall n)(a_n \geq a_{n+1})$
- *constant iff* $(\exists c)(\forall n)(a_n = c)$

Fact.

- (a_n) is nonincreasing and nondecreasing at the same time iff it is constant
- Every increasing sequence is nondecreasing and every decreasing one is nonincreasing.

Examples. Sequences (n), (2^n), (n^2) are increasing, $(\frac{1}{n})$ is decreasing, $\left((-1)^n\right)$, $(\sin n)$ are neither, nor are they nondecreasing or nonincreasing.

Arithmetic sequences

Definition.
A sequence (a_n) is called an *arithmetic sequence* iff
$(\exists d \in \mathbb{R})(\forall n = 2,3,\ldots)\ a_n - a_{n-1} = d.$

The number d is called the *difference* or the *increment* of the sequence, a_1 is sometimes called the *initial value*.

Fact. If (a_n) is an arithmetic sequence then:

- for every n, $a_n = a_1 + (n-1)d$
- if $d>0$ then (a_n) is increasing,
  if $d<0$ then (a_n) is decreasing,
  if $d=0$ then (a_n) is constant.

Theorem.

If (a_n) is an arithmetic sequence then, for every *n*,

$$a_1 + a_2 + \ ... + a_n = \sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

Proof.

Notice that for each *k*=0,1, … *n*-1, $a_{1+k} + a_{n-k} = a_1 + a_n$.

Hence, $\sum_{k=0}^{n-1}(a_{1+k}+a_{n-k})$ = $\sum_{k=0}^{n-1}(a_1+a_n) = n(a_1 + a_n)$.

But $\sum_{k=0}^{n-1}(a_{1+k}+a_{n-k}) = 2\sum_{i=1}^{n} a_i$.

OR:

$a_1 + a_2 + \ ... + a_n$ = $a_1 + (a_1 + d) + \ ... + (a_1 + (n-1)d$ = $na_1 + d\big(1 + 2 + \cdots (n-1)\big)$. Now, $1 + n - 1 = 2 + n - 2 = \ ... = n$ so $1 + 2 + \cdots (n-1) = \frac{(n-1)n}{2}$ and, finaly, $\sum_{i=1}^{n} a_i = na_1 + d\frac{(n-1)n}{2} = \frac{n}{2}\big(2a_1 + d(n-1)\big) = \frac{n}{2}\big(a_1 + a_1 + d(n-1)\big) = = \frac{n}{2}(a_1 + a_n)$. QED

Geometric sequences

Definition.
(a_n) is called a *geometric sequence (*aka *geometric progression*) iff

$$(\exists q \in \mathbb{R})[q \neq 1 \wedge (\forall n \in \mathbb{N})(a_{n+1} = q a_n)].$$

The number q is called the *common ratio* or the *quotient* of the sequence, a_1 is sometimes called the *initial value*.

Fact.

If (a_n) is a geometric sequence with the common ratio q then for every n, $a_n = a_1 q^{n-1}$.

Consider a geometric sequence (a_n). It is often convenient to number a geometric sequence starting with 0 rather than 1. Then $a_n = aq^n$ for every *n* looks better than $a_\mathrm{n} = aq^{n-1}$.

- If *a*=0 then (aq^n) is constant
  Otherwise:
- If *a*>0 and *q*>1 then (aq^n) is increasing
- If *a*<0 and *q*>1 then (aq^n) is decreasing
- If *a*>0 and 0<*q*<1 then (aq^n) is decreasing
- If *a*<0 and 0<*q*<1 then (aq^n) is increasing
- If *q*<0 then (aq^n) is called *an alternating sequence* which means that every second term of the sequence is negative and the remaining ones are positive.

Theorem.

If (a_n) is a geometric sequence with the starting term a and the quotient q then $\sum_{i=1}^{n} a_i = \frac{a(1-q^n)}{1-q}$

Proof.

$a_1 + a_2 + \ldots + a_n = a + aq + aq^2 + \cdots + aq^{n-1} =$

$a(1 + q + \cdots + \mathrm{q}^{\mathrm{n}-1}) =$ (multiply and divide by (1-*q*))

$$\frac{a(1 + q + \cdots + \mathrm{q}^{\mathrm{n}-1})(1 - \mathrm{q})}{1 - \mathrm{q}} =$$

$$\frac{a\left(1+q+\cdots+\mathrm{q}^{\mathrm{n}-1}-q-q^2-\cdots-q^n\right)}{1-\mathrm{q}} = \frac{a(1-q^n)}{1-q}.$$ QED

Remark. One reason why we do not admit 1 as the quotient of a geometric sequence is the summation formula would not work (division by zero). On the other hand, it works for sequences like (3,0,0, …) which is a geometric sequence with the starting term 3 and the quotient 0.